Determining the Singularities for the Observation of Three Image Lines
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by the complexity of the mathematical problem to solve. Therefore, it may not ensure the absence of singularity cases [4]. Additionally, even adding visual features was found in the case of three image points after rather complicated mathematical computations [2], but they are still unknown for other image features.

The evaluation of the Associate Editor and Reviewers' comments.

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Abstract

—The determination of the singularity cases in the observation of image points, can be used for interpreting geometrical loci (e.g. planes, cylinders, lines). We show that a concept named the "hidden robot", which was formerly used for understanding the singularities of a vision-based controller dedicated to parallel robots while [8] presents an approach for finding the virtual robot models associated with the Forward Geometric Model (FGM) of the parallel robots [7] but it was recently proven that it can be extended to more general cases, such as the observation of lines [11] and, thanks to this correlation, we analyze the singularities of the interaction matrix, by using advanced tools coming from the mechanical engineering community (e.g. the Grassmann-Cayley algebra [9] and/or the Grassmann geometry [10]). The interest in using these tools is that they provide the singularity cases in the observation of three image lines. We show that a concept named the "hidden robot", which proved to be efficient for finding the singularity cases in the observation of three image features is crucial in visual servoing and pose estimation. We note that these tools still require an experienced user.

Index Terms

Geometric and Kinematic Properties of Hidden Parallel Robots

In the present paper, we show that a virtual parallel robot is defined by its interaction matrix (which is the inverse of the Jacobian matrix of the robot). By kinematic property, we mean that the singularities of the inverse kinematic Jacobian matrix of the robot are the same as the singularities of the interaction matrix, by using advanced geometric interpretations of the mapping degeneracy and tools hidden within the mapping used in the observation of image lines. We show that the geometric / kinematic mapping involved in the observation of image lines is the same as the mapping that in the most complicated case where three general lines in space are observed, singularities appear when the origin of the Cartesian space. A methodology is proposed in [6] in order to define the hidden robot models associated with the time variation of the visual features and the Cartesian space. The hidden robot is indeed a tangible link between the time variation of the visual features and the singularities of the interaction matrix that defines the Cartesian space. In the other hand, the geometric property of the hidden robot is that the solutions of the virtual robot models associated with the hidden parallel robots [5], [6] are also the solutions of the 3-D local geometric primitives for estimating the pose of an object or a virtual robot in its environment. The basic idea shown in [6], [8] is that the singularity cases of a vision-based controller dedicated to parallel robots are equivalent to mappings representing the link between the time variation of the visual features and the interaction matrix. By finding this correlation, it is then possible to study the singularities of the interaction matrix by using advanced tools from the mechanical engineering community (e.g. the Grassmann-Cayley algebra [9] and/or the Grassmann geometry [10]).

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required to control a particular 3-— parallel robot. Then, in Section III, the singularities of the mapping are analyzed and validated through simulations. Finally, conclusions are drawn in Section IV.

II. THE ROBOT MODEL HIDDEN IN THE OBSERVATION OF LINES

Before presenting the architecture of robot hidden in the observation of three lines, we make some brief recalls on the computation of the related interaction matrix.

In the following section of the paper, we use the standard pin hole model with focal length equal to 1 for the representation of the camera model. However, any other model based on projective geometry could be used.

A 3-D line \( \ell_i \) of Plücker coordinates \( T^T \ell_i \) in the camera frame (being a unit vector characterizing its direction while \( u_i \), \( y_i \), \( l_i \) being the coordinates of any point \( M_i \)) belonging to \( \ell_i \) is projected in the image plane on a 2-D line \( \ell_i \) of Plücker coordinates \( T^T \ell_i \) (being a unit vector characterizing its direction while \( u_i \), \( y_i \), \( l_i \) being the coordinates of any point \( m_i \) belonging to \( \ell_i \)) (Fig. 1).

From [12], we know that

\[
\begin{bmatrix}
  u_{xi} \\
  u_{yi} \\
  u_{zi}
\end{bmatrix}
= \begin{bmatrix}
  L_{yi} / L_{xi} \\
  L_{yi} / L_{zi} \\
  L_{zi} / L_{zi}
\end{bmatrix} T^T \ell_i
\]

(1)

where \( i \), \( L_{xi}, L_{yi}, L_{zi} \) and \( \sqrt{L_{xi}} L_{zi} / L_{yi} \).

By differentiating these equations, the classical equations linking the velocities \( i \) (and as a result the velocities \( i \)) to the twist \( \tau^T_c \), \( \psi^T_c, \omega^T_c \) of the camera in its relative motion with respect to the observed object frame (\( \psi \) being the translational velocity and \( \omega \) the rotational velocity expressed in the camera frame) are

\[
i \tau_c
\]

(2)

In order to fully servo the relative motion between an object and a camera, at least three lines fixed on the object must be observed [11] (Fig. 2). Thus, considering the observation of three lines, and , the interaction matrix linking the velocities \( i \) of the lines \( \ell_i \) grouped in the vector \( T^T \tau^T  \) to the camera twist \( \tau_c \) by the relation

\[
\begin{bmatrix}
  T^T \tau^T
\end{bmatrix}^T
\]

(3)

(4)

(5)

is thus given by

\[
\begin{bmatrix}
  T^T \tau^T
\end{bmatrix}^T
\]

(7)

Singularity appears when the matrix is rank deficient. Looking at the analytical form of , it is clear that determining the singularities of through the determinant of seems to be out of reach.

Note that we have used above the Plücker representation of lines, but following results are not dependent of the choice of the representation (Cartesian, cylindrical) [12].
Joint is followed by joint rotations.

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B. Hidden robot model

In this model, we have a leg with a passive joint which is connected to the axes whose geometric properties can be parameterized, for each of them, by a passive joint by a prismatic joint whose direction is constrained to translate and to rotate around it. As a result, the vectors associated with the Plücker coordinates of the line are collinear with the axes of the cardan joint. Then, the passive joint is attached at its other extremity to a passive joint whose axis is directed along the passive 1joint by a passive cylindrical joint whose geometric properties can be parameterized, for the three observed 3-D lines, see Fig. 3.

It should be mentioned that the rotational component of the three observed 3-D lines be deduced (we need the three measures of the positions of the lines lying on a body point at the same time), see Fig. 2). This architecture is made of an active cardan (or universal joint by a prismatic joint whose axis is directed along the passive 1joint (Fig. 3). This architecture is made of an actuated parallel robot (Fig. 4), i.e. a parallel robot (PRC) architecture linking the camera frame has the same geometric and kinematic properties as the parallel robot (PRC) architecture with the camera frame as the base of the robot (PRC), respectively (Fig. 3).

The solutions of the FGM of the 3-D line measure, the orientation of the observed body single motion of a 3–U joint in this leg is necessary because we know that the location of a line is constrained to translate and to rotate around it. As a result, the vectors associated with the Plücker coordinates of the line are collinear with the axes of the cardan joint. Then, the passive joint is attached at its other extremity to a passive joint whose axis is directed along the passive 1joint by a passive cylindrical joint whose geometric properties can be parameterized, for the three observed 3-D lines, see Fig. 3.

From a mechanical engineer point of view, these geometrical properties can be parameterized, for the three observed 3-D lines, see Fig. 3. This architecture is made of an active cardan (or universal joint by a prismatic joint whose axis is directed along the passive 1joint by a passive cylindrical joint whose geometric properties can be parameterized, for the three observed 3-D lines, see Fig. 3. This architecture is made of an actuated parallel robot (Fig. 4), i.e. a parallel robot (PRC) architecture linking the camera frame has the same geometric and kinematic properties as the parallel robot (PRC) architecture with the camera frame as the base of the robot (PRC), respectively (Fig. 3).

The only information that we can extract from this measure is that the 3-D line location that we consider was attached to a rigid element (the observed body passively). Without taking into account this information, the vector pointing between the optical center and the location of the 3-D line is of dimension 3, see Fig. 2).

To prove the last item, let us consider what follows. The robot are the same as the singularities of the interaction matrix (7). As a result, the measure of the image line configuration $\ell$ is a singularity-free observation of the 3–U joint rotations $q_i$. As a result, the vectors associated with the Plücker coordinates of the line are collinear with the axes of the cardan joint. Then, the passive joint is attached at its other extremity to a passive joint whose axis is directed along the passive 1joint by a passive cylindrical joint whose geometric properties can be parameterized, for the three observed 3-D lines, see Fig. 3.
where \( \tau_c \) and \( \tau_p \) are represented by a line in the 3-D space while a pure moment will be a line in the projective space at infinity.

Then, as matrices and are never rank-deficient, we obligatorily have a loss of rank of \( \tau_c \) if and only if \( \tau_p \) is rank-deficient. Therefore, the conditions of singularity of the hidden robot inverse kinematic Jacobian matrix \( inv \) are the same as the singularity conditions of the interaction matrix \( J \).

III. SINGULARITY ANALYSIS

Singularities on the inverse Jacobian matrix \( inv \) of a parallel robot (also called or singularities [13]) appear when at least two solutions of the FGM are identical [10]. As mentioned in Section II, these singularities are analogous to the singularities of the interaction matrix.

In Type 2 singularities, parallel robots gain one (or more) uncontrollable motion. Kinematically speaking, there exists a non-null vector \( s \) defined such that \( inv \) while \( \tau_c \) i.e. the actuators are fixed (which means that \( s \) is in the null space of \( inv \)). As known in mechanics, if a rigid body got an uncontrollable motion, this means that it is not fully constrained by the system of wrenches applied on it, i.e. the static equilibrium is not ensured. As this uncontrollable motion appears only in a singularity, this means that locally the system of actuation wrenches, i.e. wrenches transmitted from the actuators to the platform by the legs, is degenerated [10].

For a given leg \( i \), any actuation wrench denoted by \( \xi_{ij} \) is reciprocal to the unit twists denoted \( \zeta_{ik} \) characterizing the displacements of the passive joints [14], i.e. \( \xi_{ij}^T \zeta_{ik} = 0 \) for any \( j \) and \( k \). This means that the virtual power developed by the wrench \( \xi_{ij} \) along the direction of motion \( \zeta_{ik} \) is null; in other words, the actuator \( j \) of the leg \( i \) cannot transmit a wrench \( \xi_{ij} \) to the platform along the direction \( \zeta_{ik} \).

Let us consider a \( UPRC \) leg belonging to our \( 3-UPRC \) hidden robot. In the frame \( i \) \( M_i, x_i, y_i, z_i \) \( (M_i \) being the point of intersection between the axis of the prismatic and revolute joints) attached to the leg, the unit twist defining the motion of the passive \( P \) joint is expressed as (see Fig. 3):

\[
\zeta_i = \begin{bmatrix}
\gamma_i \\
\gamma_i \\
\gamma_i 
\end{bmatrix}^T
\]  

(13)

while \( \zeta_i \) is the unit twist defining the motion of the passive \( R \) joint and

\[
\zeta_i = \begin{bmatrix}
\gamma_i \\
\gamma_i \\
\gamma_i 
\end{bmatrix}^T
\]

(14)

\[
\zeta_i = \begin{bmatrix}
\gamma_i \\
\gamma_i \\
\gamma_i 
\end{bmatrix}^T
\]

(15)

are the unit twists defining the motions of the passive \( C \) joint, \( \gamma_i \) being the angle between \( x_i \) and \( a_i \) (and thus \( a_{ij} \)), both axes being contained in \( c_i \) (Fig. 3).

In these twists, the three first components represent the direction of the translation velocity while the three last components represent the direction of the rotational velocity.

As a result, the unit actuation wrenches \( \xi_{ij} \) expressed in the frame \( ij \) are

\[
\xi_i = \begin{bmatrix}
\gamma_i \\
\gamma_i \\
\gamma_i 
\end{bmatrix}^T
\]

(16)

\[
\xi_i = \begin{bmatrix}
\gamma_i \\
\gamma_i \\
\gamma_i 
\end{bmatrix}^T
\]

(17)

in which \( ij \) represents the direction of the force exerted on the platform and \( ij \) the direction of the moment. As a result, \( \xi_i \) is a pure force directed along \( y_i \) and \( \xi_i \) is a pure moment reciprocal to \( \xi_i \) and \( i \) which is included in the plane \( i \) (Fig. 3).

Thus, for the three robot legs, the system of actuation wrenches is given by \( \xi \), \( \xi \), \( \xi \), \( \xi \), \( \xi \). There exists some tools able to define the conditions of degeneracy of a wrench system among which are the Grassmann geometry [10] and the Grassmann-Cayley algebra [9], [15]–[17]. In what follows, we use the Grassmann-Cayley algebra in order to find the singularity conditions related to our problem. Indeed, as any wrench of the wrench system \( \xi \) can be seen as the Plücker representation of a line\(^2\) [10], the Grassmann-Cayley algebra makes it possible to compute the determinant of the wrench system \( \xi \) as an expression involving twelve points selected on the six lines (corresponding to the six wrenches). This expression is a linear combination of 24 monomials [9], each monomial representing the volume of a tetrahedron whose extremities correspond to four of the considered twelve points. By a smart selection of the twelve points, due to the experience of the user, it is possible to vanish a large number of the monomials and thus to simplify the expression of the determinant of the wrench system.

Regarding our particular case, the Grassmann-Cayley algebra was used in [17] to prove that, if the system of wrenches is composed of a pair of three forces \( \xi \), \( \xi \), \( \xi \) and of three moments \( \xi \), \( \xi \), \( \xi \), conditions of singularities appear:

\[
f \begin{bmatrix}
\xi \\
\xi \\
\xi 
\end{bmatrix}^T = 0
\]

(18)

which means that the vectors \( \xi \), \( \xi \), and \( \xi \) (and \( \xi \), resp.) lie in the same plane (or are collinear to the same vector \( \xi \)). As a result, the mechanism is not able to resist to forces (moments, resp.) orthogonal to this plane (or in any directions orthogonal to ) and thus translations (rotations,

\(^2\)A pure force will be represented by a line in the 3-D space while a pure moment will be a line in the projective space at infinity.
resp.) are gained along the direction of (around the axis of, resp.)

From these two conditions, it is possible to find the config-
urations of the end-effector (the observed body) in the
camera frame leading to singularities. Examples of singular
configurations, depending on the arrangement of the lines on
the observed body, are detailed below.

Thanks to the particular geometric properties of the system
to analyze (invariance of robot leg configurations for any
rotation around O), it is possible to simplify the singularity
analysis by fixing the platform orientation. Thus, all expres-
sions below will be given for the “zero” platform orientation.
For another given platform orientation (parameterized by the
rotation matrix ), singularity loci will be found by parameter-
izing the variables X , Y and Z characterizing the position
of the origin of the object frame b in the camera frame
(see Figs. 5 and 6) when considering the “zero” platform
orientation thanks to new variables X , Y and Z representing
the position of the origin of the object frame for the considered
“non-zero” platform orientation such that

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}^T = \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}^T
\]  

(19)

In such a case, the three lines intersect in points M , M and
M (Fig. 5(a)) (we disregard the case where at least two lines
are parallel, which is considered in Section III-B5). We define
the frame b Q , x_b, y_b, z_b attached to the observed body
such that Q is the center of the circumsphere of the triangle
M M M and x_b is collinear to Q M. Moreover, for the
analysis, we fix the orientation of the body so that b can be
obtained from the camera frame by a translation of vector
O Q. As a result,

\[
\begin{align*}
OQ &= X Y Z^T, Q M \rho^T, \\
QM &= \rho \phi \phi T, Q M \rho \psi \psi T
\end{align*}
\]  

(20)

where ϕ, ψ are two angles defined in Fig. 5(b).

Then, from section III-A, we know that

\[
i i i O M_i , i i i i
\]  

(21)

leading to, from (18), and after some tedious developments,

\[
f Z \\
f Z X Y \rho
\]  

(22)

Condition Z means that the plane containing the lines , and
also contains the camera center O, which is not
critical in practice. In such a case, all three lines project in a
single line in the image plane and the pose of the object cannot
be determined. Condition X Y ρ means that the camera
center O lies on the cylinder whose axis is perpendicular to
the plane containing all three points M , M and M which
includes the three points. These conditions are the same as the
singularity cases where three image points are observed [2],
which is indeed not a surprise due to the equivalence of both
problems.

We define the frame b Q , x_b, y_b, z_b attached to the
observed body such that Q is the intersection point of all
three lines and x_b is collinear to (Fig. 5(b)). Again, we fix
the orientation of the body so that b can be obtained from
the camera frame by a translation of vector OQ. Thus,

\[
OQ \quad X Y Z^T, \quad T, \\
\phi \phi T, \quad \psi \psi T
\]  

(23)

where ϕ, ψ are two angles defined in Fig. 5(b).

Thus, if three coplanar lines intersect in a single point, the rank
of the interaction matrix is never full, i.e. at least one degree
of freedom (a translation towards the intersection point) of the
object cannot be controlled, as already known [11]. Moreover,
additional degrees of freedom become uncontrollable if
Z , which means as previously that the plane
containing the lines , and also contains the
camera center O.

\[
X Y \rho
\]  

which means that the camera center O lies on the line which passes through Q and which is perpendicularly to all vectors i, which is the same condition as when f .

Here, the three lines are spatial, not orthogonal and intersect
in a common point. We define the frame b Q , x_b, y_b, z_b
attached to the observed body such that x_b is collinear to
\[
OQ \quad X Y Z^T, \quad T, \\
abab T, \quad c d e T
\]  

(26)

where a, b, c, d and e are variables parameterizing the direction
of the lines.
and \( i \) can be found thanks to expression (24), and from (18) we obtain,
\[
f(b a d e Y a d b c d e X a d b c d e Z) = 0
\]
(27)

Thus, if three spatial lines intersect in a single point, the rank of the interaction matrix is never full, i.e. at least one degree of freedom (a translation) of the object cannot be controlled, as already known [11]. Moreover, additional degrees of freedom become uncontrollable if the origin of the body frame belongs to a cubic surface parameterized by \( f \).

Note also that in the case where the three lines are orthogonal, the equation \( f \) degenerates into \( XYZ \).

Here, the three lines are orthogonal and do not intersect. We define the frame \( Q, x_b, y_b, z_b \) attached to the observed body such that \( x_b \) is collinear to \( x_b \), \( y_b \) is collinear to \( y_b \), and \( z_b \) is collinear to \( z_b \). Moreover, the distance between and is equal to \( a \), between and is equal to \( b \), and between and is equal to \( c \), while \( Q \) is the barycenter of the lines (Fig. 6(a)).

As a result,
\[
\begin{align*}
OP & \quad X \ Y \ c \ Z \ a^T, \quad T \\
OP & \quad X \ b \ Y \ Z \ a^T, \quad T \\
OP & \quad X \ b \ Y \ c \ Z^T, \quad T
\end{align*}
\]
(28)

where \( P_i \) is a point belonging to \( i \).

Then, we have
\[
OP_i, \quad i \quad i \quad i \quad i \quad i
\]
(29)

which leads to, from (18),
\[
\begin{align*}
f & \quad aXY \ bYZ \ cXZ \ abc \\
f & \quad acX \ abY \ bcZ \ XYZ
\end{align*}
\]
(30)

Expression \( f \) represents a quadric surface while expression \( f \) is a cubic surface (Fig. 7).

Here, the lines \( \ell \) and \( \ell \) are parallel. We define the frame \( Q, x_b, y_b, z_b \) attached to the observed body such that \( x_b \) is collinear to \( x_b \) and \( y_b \) is lying in the plane containing and \( \ell \). In order to show the exactness of our results, we first perform simulations in the case where the three lines have a general configuration (Section III-B6). The three general
lines are parameterized as follows. We define the frame \( b, Q, x_b, y_b, z_b \) attached to the observed body such that \( x_b \) is collinear to \( y_b \). \( y_b \) is lying in the plane containing \( x_b \) and \( z_b \). Then, the line \( i \) is parameterized by its direction \( i \) and a point \( P_i \) lying on it by taking the following expressions:

\[
\begin{align*}
OP & = X b Y c Z a^T, \\
OP & = X b Y c Z a^T, \quad d e^T \tag{33}
\end{align*}
\]

where \( d, e, f, g \) and \( h \) are variables parameterizing the direction of the lines and \( a, b, c, m \) and \( X, Y, Z \) are the coordinates of \( Q \) in the camera frame. For simulation purpose, we set \( a, b, c, m \) and \( \theta, \phi, \gamma, h \). Then, a relative motion between the object frame and camera frame origins is imposed and is parameterized by the following functions characterizing the location of \( Q \) in the camera frame:

\[
\begin{align*}
X & = s, Y = s, \\
Z & = s \tag{34}
\end{align*}
\]

with \( s \) a linearly increasing function. The relative motion of the camera with respect to the observed lines in a general configuration is shown in Fig. 9. In such simulation, when \( s \), the point \( O \) reaches the configuration \( X_O \) \( m \), \( Y_O \) \( m \) and \( Z_O \) \( m \) in the object frame (see Fig. 10) which is a point lying on the cubic surface whose expression can be found in the technical report [18]. As a result, the rank-deficiency of the matrix \( M \) appears for \( s \), which is shown on Fig. 9 in which the inverse of the condition number is null only for \( s \).

A second simulation is performed in order to show now the singularities when observing three orthogonal lines in space (see Section III-B4). For simulation purpose, we take the following values of the parameters \( a, b \) and \( c \) in (28): \( a, b, m \), \( c \). Then, a relative motion between the object frame and camera frame origins is imposed and is parameterized by the following functions characterizing the location of \( Q \) in the camera frame:

\[
\begin{align*}
X & = s, Y = s, \\
Z & = s \tag{35}
\end{align*}
\]

with \( s \) a linearly increasing function. The relative motion of the camera with respect to the observed orthogonal lines is shown in Fig. 10. In such simulation, when \( s \), the point \( O \) reaches the configuration \( X_O \) \( m \), \( Y_O \) \( m \) and \( Z_O \) \( m \) in the object frame (see Fig. 10) which is a point lying on the cubic surface whose expression is given in (30). As a result, the rank-deficiency of the matrix \( M \) appears for
s = 0, which is shown on Fig. 11 in which the inverse of the condition number is null only for s = 0.

IV. CONCLUSION

In this paper, we determined the singularity cases for the observation of three image lines thanks to a tool named the “hidden robot concept”. We showed that the hidden robot concept allows for considerable simplification of the analysis, leading to the computation of new singularity cases. Indeed, the hidden robot is a tangible visualization of the mapping between the observation space and the Cartesian space. As a result, the singular configurations of the hidden robot corresponded to the singularities of the interaction matrix.

Indeed, the concept of hidden robot allowed to change the way we defined the problem. By doing so, we were able to replace the degeneracy analysis of the velocity transmission between inputs (velocity of the observed features) and outputs (camera twist), by its dual but fully equivalent problem which was to analyze the degeneracy in the transmission of wrenches between the inputs of a virtual mechanical system (virtual actuators of the hidden robot whose displacement was linked to the motions of the observed features) and its outputs (wrenches exerted on the virtual platform, i.e. the observed object).

Then, by using geometric interpretations of the mapping degeneracy and tools provided by the mechanical engineering community such as the Grassmann-Cayley algebra, we were able to find rather simple geometric interpretation of the interaction matrix degeneracy. We proved that in the most complicated cases where three general lines in space are observed, singularities appear when the origin of the observed object frame is either on a quadric or a cubic surface. In simpler cases where at least two lines belong to the same plane, these two surfaces can degenerate into simpler geometrical loci (e.g. planes, cylinders, lines).

Future works concern the analysis of the singularity cases in the observation of n lines, and research about the singularities when observing other primitives (spheres, circles, and eventually of combination of different primitives), or when using several cameras.

REFERENCES

[18] [Online]. Available: https://hal.archives-ouvertes.fr/hal-01400575